# Neighbourhoods of independence for random processes

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**Abstract**

The Freund family of distributions becomes a Riemannian 4-manifold with Fisher information as metric; we derive the induced $\alpha$-geometry, i.e., the $\alpha$-curvature, $\alpha$-Ricci curvature with its eigenvales and eigenvectors, the $\alpha$-scalar curvature etc. We show that the Freund manifold has a positive constant 0-scalar curvature, so geometrically it constitutes part of a sphere. We consider special cases as submanifolds and discuss their geometrical structures; one submanifold yields examples of neighbourhoods of the independent case for bivariate distributions having identical exponential marginals. Thus, since exponential distributions complement Poisson point processes, we obtain a means to discuss the neighbourhood of independence for random processes.

**AMS Subject Classification (2001)**: 53B1

**Key words**: Freund bivariate exponential distribution, information geometry, statistical manifold, $\alpha$-connection

## 1   Differential geometry of the Freund 4-manifold

In [4] we proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, in the subspace topology of $\mathbb{R}^3$ using information geometry and the affine immersion of Dodson and Matsuzoe [8]. For general references on information geometry, see Amari et al. [1], [2]. As part of a study of the information geometry and topology of gamma and bivariate stochastic processes cf. e.g. [5], [6], [7], we have calculated the geometry of the family of Freund bivariate mixture exponential density functions. The importance of this family lies in the fact that exponential distributions represent intervals between events for Poisson processes on the real line and Freund distributions can model bivariate processes with positive and negative covariance. The Freund family of distributions becomes a Riemannian 4-manifold with the Fisher information metric, and we derive the induced $\alpha$-geometry, i.e., the $\alpha$-Ricci curvature, the $\alpha$-scalar curvature etc. The case $\alpha = 0$ recovers the geometry of the metric or Levi Civita connection and we show that the Freund manifold has a positive constant 0-scalar curvature, so geometrically it constitutes part of a sphere.

### 1.1   Freund bivariate mixture exponential distributions

Freund [9] introduced a bivariate exponential mixture distribution arising from the following reliability considerations. Suppose that an instrument has two components $A$ and $B$ with lifetimes $X$ and $Y$ respectively having density functions (when both components are in operation)

$f_X(x) = \alpha_1\, e^{-\alpha_1 x};$

$f_Y(y) = \alpha_2\, e^{-\alpha_2 y}$

for $(\alpha_1, \alpha_2 > 0; x, y > 0)$.

Then $X$ and $Y$ are dependent in that a failure of either component changes the parameter of the life distribution of the other component. Thus when $A$ fails, the parameter for $Y$ becomes $\beta_2$; when $B$ fails, the parameter for $X$ becomes $\beta_1$. There is no other dependence. Hence the joint density function of $X$ and $Y$ is :

$$f(x,y) = \begin{cases} \alpha_1\beta_2 e^{-\beta_2 y - (\alpha_1+\alpha_2-\beta_2)x} & \text{for } 0 < x < y, \\ \alpha_2\beta_1 e^{-\beta_1 x - (\alpha_1+\alpha_2-\beta_1)y} & \text{for } 0 < y < x \end{cases} \tag{1.1}$$

where $\alpha_i, \beta_i > 0 \quad (i = 1, 2)$.

Provided that $\alpha_1 + \alpha_2 \neq \beta_1$, the marginal density function of $X$ is

$$f_X(x) \;=\; \left(\frac{\alpha_2}{\alpha_1 + \alpha_2 - \beta_1}\right) \beta_1\, e^{-\beta_1 x} + \left(\frac{\alpha_1 - \beta_1}{\alpha_1 + \alpha_2 - \beta_1}\right) (\alpha_1 + \alpha_2)\, e^{-(\alpha_1+\alpha_2)x}, \; x \geq 0 \quad (1.2)$$

and provided that $\alpha_1 + \alpha_2 \neq \beta_2$, The marginal density function of $Y$ is

$$f_Y(y) \;=\; \left(\frac{\alpha_1}{\alpha_1 + \alpha_2 - \beta_2}\right) \beta_2\, e^{-\beta_2 y} + \left(\frac{\alpha_2 - \beta_2}{\alpha_1 + \alpha_2 - \beta_2}\right) (\alpha_1 + \alpha_2)\, e^{-(\alpha_1+\alpha_2)y}, \; y \geq 0 \quad (1.3)$$

We can see that the marginal density functions are not exponential but rather mixtures of exponential distributions if $\alpha_i > \beta_i$ ; otherwise, they are weighted averages. For this reason, this system of distributions should be termed bivariate mixture exponential distributions rather than simply bivariate exponential distributions. The marginal density functions $f_X(x)$ and $f_Y(y)$ are exponential distributions only in the special case $\alpha_i = \beta_i \quad (i = 1, 2)$.

Freund discussed the statistics of the special case when $\alpha_1 + \alpha_2 = \beta_1 = \beta_2$, and obtained the joint density function as:

$$f(x,y) = \begin{cases} \alpha_1(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)y} & \text{for } 0 < x < y, \\ \alpha_2(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x} & \text{for } 0 < y < x \end{cases} \qquad (1.4)$$

with marginal density functions:

$$f_X(x) \;=\; (\alpha_1 + \alpha_2(\alpha_1 + \alpha_2)x)\, e^{-(\alpha_1+\alpha_2)x}\, x \geq 0, \qquad (1.5)$$
$$f_Y(y) \;=\; (\alpha_2 + \alpha_1(\alpha_1 + \alpha_2)y)\, e^{-(\alpha_1+\alpha_2)y}\, y \geq 0 \qquad (1.6)$$

The covariance and correlation coefficient of $X$ and $Y$ were derived by Freund, as follows:

$$Cov(X,Y) \;=\; \frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\beta_1\,\beta_2\,(\alpha_1 + \alpha_2)^2}, \qquad (1.7)$$

$$\rho(X,Y) \;=\; \frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\sqrt{\alpha_2{}^2 + 2\,\alpha_1\,\alpha_2 + \beta_1{}^2}\,\sqrt{\alpha_1{}^2 + 2\,\alpha_1\,\alpha_2 + \beta_2{}^2}} \qquad (1.8)$$

Note that $-\frac{1}{3} < \rho(X,Y) < 1$. The correlation coefficient $\rho(X,Y) \to 1$ when $\beta_1, \beta_2 \to \infty$, and $\rho(X,Y) \to -\frac{1}{3}$ when $\alpha_1 = \alpha_2$ and $\beta_1, \beta_2 \to 0$. In many applications, $\beta_i > \alpha_i \quad (i = 1, 2)$ ( i.e., lifetime tends to be shorter when the other component is out of action) ; in such cases the correlation is positive.

## 1.2   Fisher information metric

**Proposition 1.1** *Let $F$ be the set of Freund bivariate mixture exponential distributions, that is*

$$F = \{f | f(x,y; \alpha_1, \beta_1, \alpha_2, \beta_2) = \begin{cases} \alpha_1\beta_2 e^{-\beta_2 y - (\alpha_1+\alpha_2-\beta_2)x} & \text{for } 0 \leq x < y \\ \alpha_2\beta_1 e^{-\beta_1 x - (\alpha_1+\alpha_2-\beta_1)y} & \text{for } 0 \leq y \leq x \end{cases}, \; \alpha_i, \beta_i > 0\,(i,1,2)\} \; (1.9)$$

*Then we have :*

1. *Identifying $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ as a local coordinate system, $F$ can be regarded as a 4-manifold.*

2. *$F$ becomes a Riemannian manifold with the Fisher information metric $G = [g_{ij}]$ where*

$$g_{ij} = \int_0^\infty \int_0^\infty \frac{\partial^2 \log f(x,y)}{\partial x_i \partial x_j}\, f(x,y)\, dx\, dy$$

*and $(x_1, x_2, x_3, x_4) = (\alpha_1, \beta_1, \alpha_2, \beta_2)$.*
*is given by :*

$$[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1{}^2 + \alpha_1\,\alpha_2} & 0 & 0 & 0 \\ 0 & \frac{\alpha_2}{\beta_1{}^2\,(\alpha_1+\alpha_2)} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_2{}^2 + \alpha_1\,\alpha_2} & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_2{}^2\,(\alpha_1+\alpha_2)} \end{bmatrix} \qquad (1.10)$$

3. *The inverse* $[g^{ij}]$ *of* $[g_{ij}]$ *is given by :*

$$[g^{ij}] = \begin{bmatrix} \alpha_1{}^2 + \alpha_1\,\alpha_2 & 0 & 0 & 0 \\ 0 & \frac{\beta_1{}^2\,(\alpha_1+\alpha_2)}{\alpha_2} & 0 & 0 \\ 0 & 0 & \alpha_2{}^2 + \alpha_1\,\alpha_2 & 0 \\ 0 & 0 & 0 & \frac{\beta_2{}^2\,(\alpha_1+\alpha_2)}{\alpha_1} \end{bmatrix} \qquad (1.11)$$

□

## 1.3  Geometry from the $\alpha$-connection

We provide the $\alpha$-connection , and various $\alpha$-curvature objects of the Freund manifold $F$: the $\alpha$-curvature tensor, the $\alpha$-Ricci tensor, the $\alpha$-scalar curvature, the $\alpha$-sectional curvatures and the $\alpha$-mean curvatures.

1. $\alpha$-connection :

   For each $\alpha \in \mathbb{R}$, the $\alpha$ *(or* $\nabla^{(\alpha)}$ *)-connection* is the torsion-free affine connection with components:

$$\Gamma_{ij,k}^{(\alpha)} = \int_0^\infty \int_0^\infty \left( \frac{\partial^2 \log f}{\partial \xi^i \partial \xi^j} \frac{\partial \log f}{\partial \xi^k} + \frac{1-\alpha}{2} \frac{\partial \log f}{\partial \xi^i} \frac{\partial \log f}{\partial \xi^j} \frac{\partial \log f}{\partial \xi^k} \right) f \; dx \; dy$$

   **Proposition 1.2** *The functions* $\Gamma_{ij,k}^{(\alpha)}$ *are given by :*

$$\Gamma_{11,1}^{(\alpha)} = \frac{2\,(\alpha-1)\,\alpha_1 - (1+\alpha)\,\alpha_2}{2\,\alpha_1{}^2\,(\alpha_1+\alpha_2)^2},$$

$$\Gamma_{11,3}^{(\alpha)} = \frac{1+\alpha}{2\,\alpha_1\,(\alpha_1+\alpha_2)^2},$$

$$\Gamma_{12,2}^{(\alpha)} = \frac{(\alpha-1)\,\alpha_2}{2\,(\alpha_1+\alpha_2)^2\,\beta_1{}^2},$$

$$\Gamma_{13,3}^{(\alpha)} = \frac{-1+\alpha}{2\,\alpha_2\,(\alpha_1+\alpha_2)^2},$$

$$\Gamma_{14,4}^{(\alpha)} = \frac{-((\alpha-1)\,\alpha_2)}{2\,(\alpha_1+\alpha_2)^2\,\beta_2{}^2},$$

$$\Gamma_{22,2}^{(\alpha)} = \frac{(\alpha-1)\,\alpha_2}{(\alpha_1+\alpha_2)\,\beta_1{}^3},$$

$$\Gamma_{22,3}^{(\alpha)} = \frac{-((1+\alpha)\,\alpha_1)}{2\,(\alpha_1+\alpha_2)^2\,\beta_1{}^2},$$

$$\Gamma_{33,3}^{(\alpha)} = \frac{-((1+\alpha)\,\alpha_1) + 2\,(-1+\alpha)\,\alpha_2}{2\,\alpha_2{}^2\,(\alpha_1+\alpha_2)^2},$$

$$\Gamma_{34,4}^{(\alpha)} = \frac{(\alpha-1)\,\alpha_1}{2\,(\alpha_1+\alpha_2)^2\,\beta_2{}^2},$$

$$\Gamma_{44,4}^{(\alpha)} = \frac{(\alpha-1)\,\alpha_1}{(\alpha_1+\alpha_2)\,\beta_2{}^3} \qquad (1.12)$$

   *while the other independent components are zero.*                                              □

   We have an affine connection $\nabla^{(\alpha)}$ defined by :

$$\langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)},$$

So by solving the equations

$$\Gamma_{ij,k}^{(\alpha)} = \sum_{h=1}^4 g_{kh}\, \Gamma_{ij}^{h(\alpha)} \quad , (k=1,2,3,4).$$

we obtain the components of $\nabla^{(\alpha)}$ :

**Proposition 1.3** *The components* $\Gamma_{jk}^{i(\alpha)}$ *of the* $\nabla^{(\alpha)}$-*connections are given by :*

$$
\begin{aligned}
\Gamma_{11}^{(\alpha)1} &= \frac{1}{2}\left(-\frac{1+\alpha}{\alpha_1}+\frac{-1+3\,\alpha}{\alpha_1+\alpha_2}\right), \\
\Gamma_{13}^{(\alpha)1} &= \Gamma_{12}^{(\alpha)2}=\Gamma_{13}^{(\alpha)3}=\Gamma_{34}^{(\alpha)4}=\frac{-1+\alpha}{2\,(\alpha_1+\alpha_2)}, \\
\Gamma_{22}^{(\alpha)1} &= -\Gamma_{22}^{(\alpha)3}=\frac{(1+\alpha)\,\alpha_1\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_1}^2}, \\
\Gamma_{33}^{(\alpha)1} &= \Gamma_{23}^{(\alpha)2}=\frac{(1+\alpha)\,\alpha_1}{2\,\alpha_2\,(\alpha_1+\alpha_2)}, \\
\Gamma_{44}^{(\alpha)1} &= -\Gamma_{44}^{(\alpha)3}=\frac{-((1+\alpha)\,\alpha_1\,\alpha_2)}{2\,(\alpha_1+\alpha_2)\,{\beta_2}^2}, \\
\Gamma_{11}^{(\alpha)3} &= \Gamma_{14}^{(\alpha)4}=\frac{(1+\alpha)\,\alpha_2}{2\,\alpha_1\,(\alpha_1+\alpha_2)}, \\
\Gamma_{33}^{(\alpha)3} &= \frac{1}{2}\left(-\frac{1+\alpha}{\alpha_2}+\frac{-1+3\,\alpha}{\alpha_1+\alpha_2}\right), \\
\Gamma_{44}^{(\alpha)4} &= \frac{-1+\alpha}{\beta_2},
\end{aligned}
\tag{1.13}
$$

*while the other independent components are zero.* □

2. $\alpha$-Curvatures :

**Proposition 1.4** *The components* $R_{ijkl}^{(\alpha)}$ *of the* $\alpha$-*curvature tensor are given by:*

$$
\begin{aligned}
R_{1212}^{(\alpha)} &= \frac{(\alpha^2-1)\,{\alpha_2}^2}{4\,\alpha_1\,(\alpha_1+\alpha_2)^3\,{\beta_1}^2}, \\
R_{1223}^{(\alpha)} &= \frac{(\alpha^2-1)\,\alpha_2}{4\,(\alpha_1+\alpha_2)^3\,{\beta_1}^2}, \\
R_{1414}^{(\alpha)} &= \frac{(\alpha^2-1)\,\alpha_2}{4\,(\alpha_1+\alpha_2)^3\,{\beta_2}^2}, \\
R_{1434}^{(\alpha)} &= \frac{-(\alpha^2-1)\,\alpha_1}{4\,(\alpha_1+\alpha_2)^3\,{\beta_2}^2}, \\
R_{2323}^{(\alpha)} &= \frac{(\alpha^2-1)\,\alpha_1}{4\,(\alpha_1+\alpha_2)^3\,{\beta_1}^2}, \\
R_{2424}^{(\alpha)} &= \frac{(\alpha^2-1)\,\alpha_1\,\alpha_2}{4\,(\alpha_1+\alpha_2)^2\,{\beta_1}^2\,{\beta_2}^2}, \\
R_{3434}^{(\alpha)} &= \frac{(\alpha^2-1)\,{\alpha_1}^2}{4\,\alpha_2\,(\alpha_1+\alpha_2)^3\,{\beta_2}^2},
\end{aligned}
\tag{1.14}
$$

*while the other independent components are zero.* □

Contracting $R_{ijkl}^{(\alpha)}$ with $g^{il}$ we obtain the components $R_{jk}^{(\alpha)}$ of the $\alpha$-Ricci tensor.

**Proposition 1.5** *The* $\alpha$-*Ricci tensor* $R^{(\alpha)}=[R_{ij}^{(\alpha)}]$ *is given by :*

$$
R^{(\alpha)}=[R_{jk}^{(\alpha)}]=\begin{bmatrix}
\frac{-(\alpha^2-1)\,\alpha_2}{2\,\alpha_1\,(\alpha_1+\alpha_2)^2} & 0 & \frac{\alpha^2-1}{2\,(\alpha_1+\alpha_2)^2} & 0 \\
0 & \frac{-(\alpha^2-1)\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_1}^2} & 0 & 0 \\
\frac{\alpha^2-1}{2\,(\alpha_1+\alpha_2)^2} & 0 & \frac{-(\alpha^2-1)\,\alpha_1}{2\,\alpha_2\,(\alpha_1+\alpha_2)^2} & 0 \\
0 & 0 & 0 & \frac{-(\alpha^2-1)\,\alpha_1}{2\,(\alpha_1+\alpha_2)\,{\beta_2}^2}
\end{bmatrix}
\tag{1.15}
$$

*The $\alpha$-eigenvalues and the $\alpha$-eigenvectors of the $\alpha$-Ricci tensor are given by :*

$$\left(\alpha^2 - 1\right) \begin{pmatrix} 0 \\ \frac{1}{(\alpha_1+\alpha_2)^2} - \frac{1}{2\,\alpha_1\,\alpha_2} \\ \frac{-\alpha_2}{2\,(\alpha_1+\alpha_2)\,\beta_1{}^2} \\ \frac{-\alpha_1}{2\,(\alpha_1+\alpha_2)\,\beta_2{}^2} \end{pmatrix} \tag{1.16}$$

$$\begin{pmatrix} \frac{\alpha_1}{\alpha_2} & 0 & 1 & 0 \\ -\frac{\alpha_2}{\alpha_1} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.17}$$

$\square$

**Proposition 1.6** *The manifold $F$ has a constant $\alpha$-scalar curvature*

$$R^{(\alpha)} = \frac{-3\left(\alpha^2 - 1\right)}{2} \tag{1.18}$$

*Note that the Freund manifold has a positive scalar curvature $R^{(0)} = \frac{3}{2}$ when $\alpha = 0$. So geometrically it constitutes part of sphere.* $\square$

**Proposition 1.7** *The $\alpha$-sectional curvatures $\varrho^{(\alpha)}(\lambda, \mu)\,(\lambda, \mu = 1, 2, 3, 4)$ are given by :*

$$\begin{aligned} \varrho^{(\alpha)}(1, 2) &= \varrho^{(\alpha)}(1, 4) = \frac{\left(1 - \alpha^2\right)\alpha_2}{4\,(\alpha_1 + \alpha_2)}, \\ \varrho^{(\alpha)}(1, 3) &= 0, \\ \varrho^{(\alpha)}(2, 3) &= \frac{\left(1 - \alpha^2\right)\alpha_1}{4\,(\alpha_1 + \alpha_2)}, \\ \varrho^{(\alpha)}(2, 4) &= \frac{1 - \alpha^2}{4}, \\ \varrho^{(\alpha)}(3, 4) &= \varrho(2, 3). \end{aligned} \tag{1.19}$$

$\square$

**Proposition 1.8** *The $\alpha$-mean curvatures $\varrho^{(\alpha)}(\lambda)\,(\lambda = 1, 2, 3, 4)$ are given by :*

$$\begin{aligned} \varrho^{(\alpha)}(1) &= \frac{\left(1 - \alpha^2\right)\alpha_2}{6\,(\alpha_1 + \alpha_2)}, \\ \varrho^{(\alpha)}(2) &= \varrho(4) = \frac{1 - \alpha^2}{6}, \\ \varrho^{(\alpha)}(3) &= \frac{\left(1 - \alpha^2\right)\alpha_1}{6\,(\alpha_1 + \alpha_2)}. \end{aligned} \tag{1.20}$$

$\square$

# 2 Submanifolds of the Freund 4-manifold

We consider five submanifolds $F_i\,(i = 1, 2, 3, 4)$ of the 4-manifold $F$ of Freund bivariate exponential distributions $f(x, y; \alpha_1, \beta_1, \alpha_2, \beta_2)$ (1.1), which includes the case of statistically independent random variables. It includes also the special case of an Absolutely Continuous Bivariate Exponential Distribution called ACBED (or ACBVE) by Block and Basu (cf. Hutchinson and Lai [10]). We use the coordinate system $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ for the submanifolds $F_i\,(i \neq 3)$, and the coordinate system $(\lambda_1, \lambda_{12}, \lambda_2)$ for ACBED of the Block and Basu case.

## 2.1  Submanifold $F_1 \subset F$ : $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$

The distributions are of form :

$$f(x, y; \alpha_1, \alpha_2) = f_1(x; \alpha_1) f_2(y; \alpha_2) \tag{2.21}$$

where $f_i$ are the density functions of the one-dimensional exponential distributions with the parameters $\alpha_i > 0 \, (i = 1, 2)$. This is the case for statistical independence of $X$ and $Y$, so the space $F_1$ is the direct product of two Riemannian spaces $\{f_1(x; \alpha_1) : f_1(x; \alpha_1) = \alpha_1 e^{-\alpha_1 x}, \, \alpha_1 > 0\}$ and $\{f_2(y; \alpha_2) : f_2(y; \alpha_2) = -\alpha_2 e^{-\alpha_2 y}, \, \alpha_2 > 0\}$.

**Proposition 2.1**  *The metric tensor $[g_{ij}]$ is as follows :*

$$[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix} \tag{2.22}$$

□

**Proposition 2.2**  *The components of the $\alpha$-connection are*

$$\begin{aligned}
\Gamma_{11,1}^{(\alpha)} &= \frac{\alpha - 1}{\alpha_1^{\,3}}, \\
\Gamma_{22,2}^{(\alpha)} &= \frac{\alpha - 1}{\alpha_2^{\,3}}, \\
\Gamma_{11}^{(\alpha)1} &= \frac{\alpha - 1}{\alpha_1}, \\
\Gamma_{22}^{(\alpha)2} &= \frac{\alpha - 1}{\alpha_2},
\end{aligned} \tag{2.23}$$

*while the other components are zero.*                                                                 □

**Proposition 2.3**  *The $\alpha$-curvature tensor, $\alpha$-Ricci tensor, and $\alpha$-scalar curvature of $F_1$ are zero.*  □

## 2.2  Submanifold $F_2 \subset F$ : $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$

The distributions are of form :

$$f(x, y; \alpha_1, \beta_1) = \begin{cases} \alpha_1 \beta_1 \, e^{-\beta_1 y - (2\alpha_1 - \beta_1)x} & \text{for } 0 < x < y \\ \alpha_1 \beta_1 \, e^{-\beta_1 x - (2\alpha_1 - \beta_1)y} & \text{for } 0 < y < x \end{cases} \tag{2.24}$$

with parameters $\alpha_1, \beta_1 > 0$. The covariance, correlation coefficient and marginal density functions, of $X$ and $Y$ are given by :

$$\begin{aligned}
Cov(X, Y) &= \frac{1}{4} \left( \frac{1}{\alpha_1^{\,2}} - \frac{1}{\beta_1^{\,2}} \right), \tag{2.25} \\
\rho(X, Y) &= 1 - \frac{4\,\alpha_1^{\,2}}{3\,\alpha_1^{\,2} + \beta_1^{\,2}}, \tag{2.26} \\
f_X(x) &= \left( \frac{\alpha_1}{2\,\alpha_1 - \beta_1} \right) \beta_1 \, e^{-\beta_1 x} + \left( \frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1} \right) (2\,\alpha_1) \, e^{-2\,\alpha_1 x}, \, x \geq 0, \tag{2.27} \\
f_Y(y) &= \left( \frac{\alpha_1}{2\,\alpha_1 - \beta_1} \right) \beta_1 \, e^{-\beta_1 y} + \left( \frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1} \right) (2\,\alpha_1) \, e^{-2\,\alpha_1 y}, \, y \geq 0. \tag{2.28}
\end{aligned}$$

Note that $\rho(X, Y) = 0$ when $\alpha_1 = \beta_1$.

Note that $F_2$ forms as exponential family, with parameters $(\alpha_1, \beta_1)$ and potential function

$$\psi = -\log(\alpha_1 \beta_1) \tag{2.29}$$

**Proposition 2.4**  *The submanifold $F_2$ is an isometric isomorph of a the manifold $F_1$.*

**Proof:** Since $\psi$ is a potential function, the Fisher metric is given by the Hessian of $\psi$, that is,

$$g_{ij} = \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}. \tag{2.30}$$

Then, we have the Fisher metric (2.22) by a straightforward calculation. $\qquad\square$

### 2.2.1 Mutually dual foliations:

Since $\nabla^{(1)}_{\partial_i} \partial_j = 0$, $(\alpha_1, \beta_1)$ is a 1-affine coordinate system, and the (-1)-affine coordinate system is given by:

$$\eta_1 = -\frac{1}{\alpha_1},$$
$$\eta_2 = -\frac{1}{\beta_1}. \tag{2.31}$$

These coordinate have potential function:

$$\lambda = \log(\alpha_1 \, \beta_1) - 2. \tag{2.32}$$

So the coordinates $(\alpha_1, \beta_1)$ and $(-\frac{1}{\alpha_1}, -\frac{1}{\beta_1})$ are mutually dual with respect to the Fisher metric, and the tetrad $\{F_2, g, \nabla^{(1)}, \nabla^{(-1)}\}$ is dually flat space. Therefore $F_2$ has dually orthogonal foliations. For example: take $(\alpha_1, \eta_2)$ as a coordinate system for $F_2$; then

$$f(x, y; \alpha_1, \eta_2) = \begin{cases} -\frac{\alpha_1}{\eta_2} e^{\left(\frac{1}{\eta_2}\right)y - \left(2\,\alpha_1 + \frac{1}{\eta_2}\right)x} & \text{for } 0 < x < y \\ -\frac{\alpha_1}{\eta_2} e^{\left(\frac{1}{\eta_2}\right)x - \left(2\,\alpha_1 + \frac{1}{\eta_2}\right)y} & \text{for } 0 < y < x \end{cases} \tag{2.33}$$

and the Fisher metric is :

$$[g_{ij}] = \begin{bmatrix} -\frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{(\eta_2)^2} \end{bmatrix} \tag{2.34}$$

### 2.2.2 Neighbourhoods of independence in $F_2$

An important practical application of the Freund submanifold $F_2$ is the representation of a bivariate stochastic proces for which the marginals are identical exponentials. The next result is important because it provides topological neighbourhoods of that subspace $W$ in $F_2$ consisting of the bivariate processes that have zero covariance: we obtain neighbourhoods of independence for random (ie exponentially distributed) processes.

**Proposition 2.5** *Let $\{F_2, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be the manifold $F_2$ with Fisher metric $g$ and exponential connection $\nabla^{(1)}$. Then $F_2$ can be realized in Euclidean $\mathbb{R}^3$ by the graph of a potential function, namely, $F_2$ can be realized by the affine immersion $\{h, \xi\}$:*

$$h : \mathcal{G} \to \mathbb{R}^3 : \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \psi \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

*where $\psi = -\log(\alpha_1 \, \beta_1)$ and $\xi$ is the transversal vector field along $h$.*

*In $F_2$, the submanifold $W$ consisting of the independent case $(\alpha_1 = \beta_1)$ is represented by the curve :*

$$(0, \infty) \to \mathbb{R}^3 : (\alpha_1) \mapsto (\alpha_1, -\log(\alpha_1 \, \beta_1)), \quad \xi = (0, 0, 1).$$

This is illustrated in the graphic which shows $S$, an affine embedding of $F_2$ as a surface in $\mathbb{R}^3$, and $T$ an $\mathbb{R}^3$-tubular neighbourhood of $W$, the curve $\alpha_1 = \beta_1$ in the surface. This curve $W$ represents all bivariate distributions having identical exponential marginals and zero covariance; its tubular neighourhood $T$ represents departures from independence.
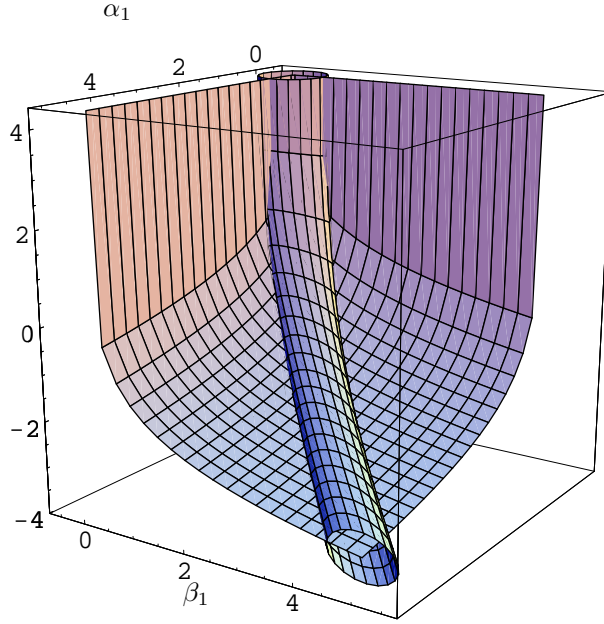
Figure 1: *Affine immersion in natural coordinates $(\alpha_1, \beta_1)$ as a surface in $\mathbb{R}^3$ for the Freund submanifold $F_2$; the tubular neighbourhood surrounds the curve $(\alpha_1 = \beta_1$ in the surface) consisting of all bivariate distributions having identical exponential marginals and zero covariance.*

## 2.3  Submanifold $F_3 \subset F$: $\beta_1 = \beta_2 = \alpha_1 + \alpha_2$

The distributions are of form:

$$f(x, y; \alpha_1, \alpha_2, \beta_2) = \begin{cases} \alpha_1 \, (\alpha_1 + \alpha_2) \, e^{-(\alpha_1 + \alpha_2)y} & \text{for } 0 < x < y \\ \alpha_2 \, (\alpha_1 + \alpha_2) \, e^{-(\alpha_1 + \alpha_2)x} & \text{for } 0 < y < x \end{cases} \qquad (2.35)$$

with parameters $\alpha_1, \alpha_2 > 0$. The covariance, correlation coefficient and marginal functions, of $X$ and $Y$ are given by :

$$Cov(X, Y) = \frac{{\alpha_1}^2 + \alpha_1 \, \alpha_2 + {\alpha_2}^2}{(\alpha_1 + \alpha_2)^4}, \qquad (2.36)$$

$$\rho(X, Y) = \frac{{\alpha_1}^2 + \alpha_1 \, \alpha_2 + {\alpha_2}^2}{\sqrt{2 \, (\alpha_1 + \alpha_2)^2 - {\alpha_1}^2} \, \sqrt{2 \, {\alpha_1}^2 + 4 \, \alpha_1 \alpha_2 + {\alpha_2}^2}}, \qquad (2.37)$$

$$f_X(x) = (\alpha_2 \, (\alpha_1 + \alpha_2)x + \alpha_1) \, e^{-(\alpha_1 + \alpha_2)x}, \; x \geq 0 \qquad (2.38)$$

$$f_Y(y) = (\alpha_1 \, (\alpha_1 + \alpha_2)y + \alpha_2) \, e^{-(\alpha_1 + \alpha_2)y}, \; y \geq 0 \qquad (2.39)$$

Note that the correlation coefficient is positive.

**Proposition 2.6** *The metric tensor $[g_{ij}]$ is given by :*

$$[g_{ij}] = \begin{bmatrix} \frac{\alpha_2 + 2 \, \alpha_1}{\alpha_1 \, (\alpha_1 + \alpha_2)^2} & \frac{1}{(\alpha_1 + \alpha_2)^2} \\ \frac{1}{(\alpha_1 + \alpha_2)^2} & \frac{\alpha_1 + 2 \, \alpha_2}{\alpha_2 \, (\alpha_1 + \alpha_2)^2} \end{bmatrix} \qquad (2.40)$$

$\square$

**Proposition 2.7** *The components of the $\alpha$-connection of $F_3$ are*

$$\Gamma_{11}^{(\alpha)1} = \frac{-\left(\frac{1+\alpha}{\alpha_1}\right) + \frac{-1+3\alpha}{\alpha_1+\alpha_2}}{2},$$

$$\Gamma_{12}^{(\alpha)1} = \frac{-1+\alpha}{2\left(\alpha_1+\alpha_2\right)},$$

$$\Gamma_{22}^{(\alpha)1} = \frac{(1+\alpha)\,\alpha_1}{2\,\alpha_2\,\left(\alpha_1+\alpha_2\right)},$$

$$\Gamma_{11}^{(\alpha)2} = \frac{(1+\alpha)\,\alpha_2}{2\,\alpha_1\,\left(\alpha_1+\alpha_2\right)},$$

$$\Gamma_{22}^{(\alpha)2} = \frac{-\left(\frac{1+\alpha}{\alpha_2}\right) + \frac{-1+3\alpha}{\alpha_1+\alpha_2}}{2}, \tag{2.41}$$

*while the other independent components are zero.* □

**Proposition 2.8** *The $\alpha$-curvature tensor, $\alpha$-Ricci curvature, and $\alpha$-scalar curvature of $F_3$ are zero.*

## 2.4   Submanifold $F_4 \subset F$: ACBED of Block and Basu

Consider the distributions are form:

$$f(x, y; \lambda_1, \lambda_{12}, \lambda_2) = \begin{cases} \frac{\lambda_1\,\lambda\,(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2}\,e^{-\lambda_1\,x-(\lambda_2+\lambda_{12})\,y} & \text{for } 0 < x < y \\ \frac{\lambda_2\,\lambda\,(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2}\,e^{-(\lambda_1+\lambda_{12})\,x-\lambda_2\,y} & \text{for } 0 < y < x \end{cases} \tag{2.42}$$

where the parameters $\lambda_1, \lambda_{12}, \lambda_2$ are positive, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.
This distribution was derived originally by omitting the singular part of the Marshall and Olkin distribution (cf. [11], page [139]); Block and Basu called it the ACBED to emphasize that they are the absolutely continuous bivariate exponential distributions. Alternatively, it can be derived by Freund's method (1.1), with

$$\alpha_1 = \lambda_1 + \frac{\lambda_1\,\lambda_{12}}{(\lambda_1+\lambda_2)},$$

$$\beta_1 = \lambda_1 + \lambda_{12},$$

$$\alpha_2 = \lambda_2 + \frac{\lambda_2\,\lambda_{12}}{(\lambda_1+\lambda_2)},$$

$$\beta_2 = \lambda_2 + \lambda_{12}.,$$

By substitutions we obtained the covariance, correlation coefficient and marginal functions:

$$Cov(X,Y) = \frac{(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12}) - \lambda^2\,\lambda_1\,\lambda_2}{\lambda^2\,(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12})}, \tag{2.43}$$

$$\rho(X,Y) = \frac{(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12}) - \lambda^2\,\lambda_1\,\lambda_2}{\sqrt{\prod_{i=1,\,j\neq i}^{2}\left((\lambda_1+\lambda_2)^2(\lambda_i+\lambda_{12})^2 + \lambda_j\lambda^2\,(\lambda_j+2\lambda_i)\right)}}, \tag{2.44}$$

$$f_X(x) = \left(\frac{-\lambda_{12}}{\lambda_1+\lambda_2}\right)\lambda\,e^{-\lambda\,x} + \left(\frac{\lambda}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_{12})\,e^{-(\lambda_1+\lambda_{12})\,x},\ x \geq 0 \tag{2.45}$$

$$f_Y(y) = \left(\frac{-\lambda_{12}}{\lambda_1+\lambda_2}\right)\lambda\,e^{-\lambda\,y} + \left(\frac{\lambda}{\lambda_1+\lambda_2}\right)(\lambda_2+\lambda_{12})\,e^{-(\lambda_2+\lambda_{12})\,y},\ y \geq 0 \tag{2.46}$$

Note that the correlation coefficient is positive, and the marginal functions are a negative mixture of two exponentials.

**Proposition 2.9** *The metric tensor $[g_{ij}]$ using the coordinate system $(\lambda_1, \lambda_{12}, \lambda_2)$ is as follows:*

$$[g_{ij}] = \begin{bmatrix} \dfrac{\lambda_2\left(\frac{1}{\lambda_1} + \frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \dfrac{\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \dfrac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \\[2ex] \dfrac{\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \dfrac{\frac{\lambda_2}{(\lambda_1+\lambda_{12})^2} + \frac{\lambda_1}{(\lambda_2+\lambda_{12})^2}}{\lambda_1+\lambda_2} + \frac{1}{\lambda^2} & \dfrac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} \\[2ex] \dfrac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \dfrac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} & \dfrac{\lambda_1\left(\frac{1}{\lambda_2} + \frac{\lambda_1+\lambda_2}{(\lambda_2+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \end{bmatrix} \qquad (2.47)$$

$\square$

The Christoffel symbols, curvature tensor, Ricci tensor, scalar curvature, sectional curvatures and the mean curvatures were computed but are not listed because they are somewhat cumbersome.

In the case when $\lambda_1 = \lambda_2$, this family of distributions becomes

$$f(x, y; \lambda_1, \lambda_{12}) = \begin{cases} \dfrac{(2\lambda_1+\lambda_{12})(\lambda_1+\lambda_{12})}{2} e^{-\lambda_1 x - (\lambda_1+\lambda_{12})y} & \text{for } 0 < x < y \\[2ex] \dfrac{(2\lambda_1+\lambda_{12})(\lambda_1+\lambda_{12})}{2} e^{-\lambda_1 y - (\lambda_1+\lambda_{12})x} & \text{for } 0 < y < x \end{cases} \qquad (2.48)$$

which is an exponential family with natural parameters $(\theta_1, \theta_2) = (\lambda_1, \lambda_{12})$ and potential function $\psi(\theta) = \log(2) - \log(\lambda_1 + \lambda_{12}) - \log(2\lambda_1 + \lambda_{12})$, note that from equations (2.45, 2.46), this family of distributions has an identical marginal density functions.

So it would be easy to derive the $\alpha$ -geometry, for example:

The metric tensor $[g_{ij}]$ is as follows:

$$[g_{ij}] = \begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{4}{(2\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{2}{(2\lambda_1+\lambda_{12})^2} \\[1.5ex] \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{2}{(2\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{1}{(2\lambda_1+\lambda_{12})^2} \end{bmatrix} \qquad (2.49)$$

By direct calculation the functions $\Gamma_{ij,k}^{(\alpha)} = \frac{1-\alpha}{2} \partial_i \partial_j \partial_k \psi(\theta)$; are given by:

$$\begin{aligned} \Gamma_{11,1}^{(\alpha)} &= (1-\alpha)\left(\frac{-1}{(\lambda_1+\lambda_{12})^3} - \frac{8}{(2\lambda_1+\lambda_{12})^3}\right), \\[1.5ex] \Gamma_{11,2}^{(\alpha)} &= (1-\alpha)\left(\frac{-1}{(\lambda_1+\lambda_{12})^3} - \frac{4}{(2\lambda_1+\lambda_{12})^3}\right), \\[1.5ex] \Gamma_{12,2}^{(\alpha)} &= (1-\alpha)\left(\frac{-1}{(\lambda_1+\lambda_{12})^3} - \frac{2}{(2\lambda_1+\lambda_{12})^3}\right), \\[1.5ex] \Gamma_{22,2}^{(\alpha)} &= (1-\alpha)\left(\frac{-1}{(\lambda_1+\lambda_{12})^3} - \frac{1}{(2\lambda_1+\lambda_{12})^3}\right). \end{aligned} \qquad (2.50)$$

By solving the equations

$$\Gamma_{ij,k}^{(\alpha)} = \sum_{h=1}^{3} g_{kh}\, \Gamma_{ij}^{h(\alpha)} \quad , (k = 1, 2)$$

we obtain the components of $\nabla^{(\alpha)}$ as follows:

$$\Gamma^{(\alpha)1} = [\Gamma_{ij}^{(\alpha)1}] = \begin{bmatrix} \frac{1-\alpha}{\lambda_1+\lambda_{12}} + \frac{4(\alpha-1)}{2\lambda_1+\lambda_{12}} & \frac{(\alpha-1)\lambda_{12}}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} \\[1.5ex] \frac{(\alpha-1)\lambda_{12}}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} & \frac{-(\alpha-1)\lambda_1}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} \end{bmatrix},$$

$$\Gamma^{(\alpha)2} = [\Gamma_{ij}^{(\alpha)2}] = \begin{bmatrix} \frac{-2(\alpha-1)\lambda_{12}}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} & \frac{2(\alpha-1)\lambda_1}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} \\[1.5ex] \frac{2(\alpha-1)\lambda_1}{(\lambda_1+\lambda_{12})(2\lambda_1+\lambda_{12})} & \frac{2(\alpha-1)}{\lambda_1+\lambda_{12}} + \frac{1-\alpha}{2\lambda_1+\lambda_{12}} \end{bmatrix}.$$

In this case, the $\alpha$-curvature tensor, $\alpha$-Ricci curvature, and $\alpha$-scalar curvature are zero.

In addition, since the coordinates $(\lambda_1, \lambda_{12})$ is a 1-affine coordinate system, then (-1)-affine coordinate system is

$$(\eta_1, \eta_2) = (-\frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{\lambda_1 + 2\,\lambda_{12}}, -\frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{2\,\lambda_1 + \lambda_{12}})$$

with potential function

$$\lambda = -2 - \log(2) + \log(2\,\lambda_1 + \lambda_{12}) + \log(\lambda_1 + \lambda_{12}).$$

## 3 Freund bivariate mixture log-exponential distributions

In this section we introduce a Freund bivariate mixture log-exponential distribution which has mixture log-exponential marginal functions, and discus their properties.

The Freund bivariate mixture log-exponential distributions arise from the Freund distributions (1.1)for the non-negative random variables $x = log\frac{1}{n}$ and $y = log\frac{1}{m}$, or equivalently, $n = e^{-x}$ and $m = e^{-y}$. So the Freund log-exponential distributions are given by:

$$g(n,m) = \begin{cases} \alpha_1\,\beta_2\,m^{(\beta_2-1)}\,n^{(\alpha_1+\alpha_2-\beta_2-1)} & \text{for } 0 < m < n < 1, \\ \alpha_2\,\beta_1\,n^{(\beta_1-1)}\,m^{(\alpha_1+\alpha_2-\beta_1-1)} & \text{for } 0 < n < m < 1 \end{cases} \tag{3.51}$$

where $\alpha_i, \beta_i > 0 \quad (i = 1, 2)$. The covariance, and marginal density functions, of $n$ and $m$ are given by:

$$Cov(n,m) = \frac{\alpha_2\,(-(\alpha_1\,(2+\alpha_1+\alpha_2))+\beta_1) + (\alpha_1 + (\alpha_1+\alpha_2)\,\beta_1)\,\beta_2}{(1+\alpha_1+\alpha_2)^2\,(2+\alpha_1+\alpha_2)\,(1+\beta_1)\,(1+\beta_2)}, \tag{3.52}$$

$$g_N(n) = \left(\frac{\alpha_2}{\alpha_1+\alpha_2-\beta_1}\right)\beta_1 n^{\beta_1-1} + \left(\frac{\alpha_1-\beta_1}{\alpha_1+\alpha_2-\beta_1}\right)(\alpha_1+\alpha_2)n^{(\alpha_1+\alpha_2)-1}, \tag{3.53}$$

$$g_M(m) = \left(\frac{\alpha_1}{\alpha_1+\alpha_2-\beta_2}\right)\beta_2 m^{\beta_2-1} + \left(\frac{\alpha_2-\beta_2}{\alpha_1+\alpha_2-\beta_2}\right)(\alpha_1+\alpha_2)m^{(\alpha_1+\alpha_2)-1}. \tag{3.54}$$

Note that the marginal functions are mixture log-exponential distributions. Directly from the definition of the Fisher metric we deduce:

**Proposition 3.1** *The family of Freund bivariate mixture log-exponential distributions for random variables $n, m$ determines a Riemannian 4-manifold which is an isometric isomorph of the Freund 4-manifold .*

## 4 Concluding remarks

We have derived the information geometry of the 4-manifold of Freund bivariate mixture exponential distributions, which admits positive and negative covariance. The curvature objects are derived and so also are those on four submanifolds, including the case of statistically independent random variables, and the special case ACBED of Block and Basu. We use one submanifold to provide examples of neighbourhoods of the independent case for bivariate distributions having identical exponential marginals. Thus, since exponential distributions complement Poisson point processes, we obtain a means to discuss the neighbourhood of independence for random processes in general. The Freund manifold has a constant 0-scalar curvature, so geometrically it constitutes part of a sphere.

The authors used *Mathematica* to perform analytic calculations [3], and can make available working notebooks for others to use.

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